SS 2017

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Manifolds and Group actions

Homework 10

Mandatory Exercise 1. (6 Points)
Let $i : S^n \to \mathbb{R}^{n+1}$ be the inclusion. In this exercise we consistently view $T_pS^n \subset \mathbb{R}^{n+1}$. We orient S^n and $\mathbb{R}^{n+1}$ in the standard way.

a) Let X be a vector field on S^n that does not vanish. Show that the map

$$H(t,p) = \cos(\pi t)p + \sin(\pi t)\frac{X_p}{\|X_p\|}.$$

is a homotopy between the identity map and the antipodal map.

b) Let ω be the $(n-1)$ form on $\mathbb{R}^{n-1}$ given by

$$\omega = \sum_{i=1}^{n+1}(-1)^{i+1}x^i dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^{n+1}$$

Compute $d\omega$ and show that $\int_{S^n} i^*\omega > 0$.

c) Show that the vector field X cannot exist if n is even. What about odd n?

Mandatory Exercise 2. (14 Points)
Let M, N be closed oriented manifolds of dimension n. Let $f : M \to N$ be a smooth map. It can be shown that there exists a real number $\deg(f)$ such that, for all $\omega \in \Omega^n(N)$ the following equality holds

$$\int_M f^*\omega = \deg(f)\int_N \omega.$$

a) Show that if f is not surjective, then $\deg(f) = 0$.

b) Show that if f is an orientation preserving diffeomorphism, that then $\deg(f) = 1$, and that if f is an orientation reversing diffeomorphism, that then $\deg(f) = -1$.

c) Let $f : M \to N$ be surjective, and let q be a regular value for f. Show that $f^{-1}(q)$ is a finite set and that there exists a neighborhood W of q in N such that $f^{-1}(W)$ is a disjoint union of open sets V_i of M with $f\big|_{V_i} : V_i \to W$ is a diffeomorphism.

d) Suppose that f has a regular value. Show that $\deg(f)$ is an integer. *Remark: One can show that in this setting the regular values of f are open and dense. Any map has plenty of regular values and the degree is always an integer.*

e) Given $k \in \mathbb{Z}$, construct a smooth map $f_k : S^1 \to S^1$ of degree k.

f) Show that homotopic maps have the same degree.

g) Let $f : S^n \to S^n$ be an orientation preserving diffeomorphism if n is even, or an orientation-reversing diffeomorphism if n is odd. Prove that f must have a fixed point. *Hint: If f has no fixed points, you can construct a homotopy between f and the antipodal map.*

Hand in on 3th of July in the pigeonhole on the third floor.